# /dev/joe Crescents and Vortices

by Joseph DeVincentis

## Introduction

Erich Friedman posed this problem on his Math Magic web site in March 2012[1]: Find the shape of largest area such that N of them, in total, fit into two different shapes. In one variation of the puzzle, the shapes were circles of diameters 2 and 3, and once N was sufficiently large, the solutions that people (including Maurizio Morandi, Jeremy Galvagni, Andrew Bayly, and myself) were finding tended to be concave shapes that nested together in two different ways to fit inside the two circles. I discovered an interesting series of shapes of this sort which I named */dev/joe crescents*. It appears that Bayly's solution for N=9 is in fact one type of /dev/joe crescent, but I generalized the type and built many more solutions on this model.

Since they are meant to fit into circles, it is natural that their edges consist entirely circular arcs. This makes them relatives of Reuleaux polygons. In Reuleaux polygons, these arcs are centered at the vertices, a property which makes them have constant width. In /dev/joe crescents, the arcs are not centered at the vertices, but are determined in quite a different way, which leads to some very odd looking shapes.

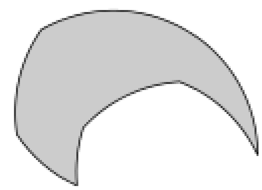

*Fig. 1: A single /dev/joe crescent of type 1 for S=3, L=7*

The governing rule that determines this shape is that they fit together, rotated with respect to one another, inside a circle. Although the shape above has 6 distinct arcs, I think of it as having 3 edges: the outside (the single arc which is much longer than the rest), the leading edge (the other 2 arcs on the convex side), and the trailing edge (the 3 arcs on the concave side).

In this shape, which I call a type 1 crescent, all of the arcs are from circles of radius 1. The three edges which appear to be the same size are in fact the same size. Four of the angles between arcs which (two inner and two outer) are the same. S=3 indicates that three of these fit inside a circle of diameter 2, so the outside edge is 1/3 of a circle.

One arc on the leading edge is the same length as two of the arcs on the trailing edge. That one arc can be aligned against either of the other two. One way, it takes S=3 pieces to go around a small circle with all of the outside edges forming that circle. The other way, L=7 indicates that it takes 7 of these wrap around and meet up with the first shape to make a circular pattern, with each outside edge tangent to the circle.

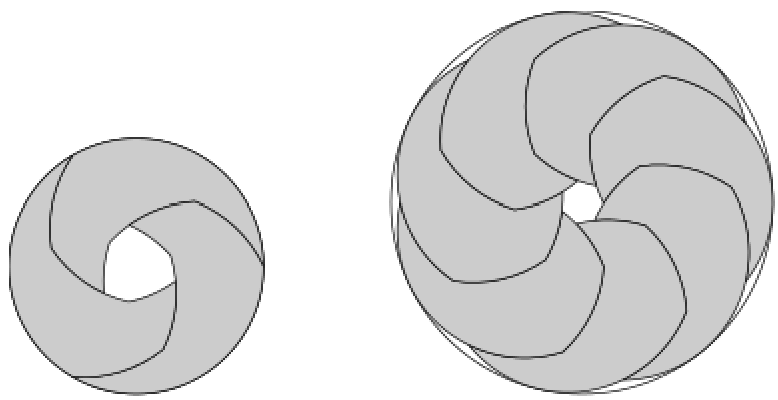

*Fig. 2: /dev/joe vortex of type 1 for S=3, L=7*

This creates two striking (and strikingly different) patterns. The small circle resembles a sort of vortex of fluid swirling into the central hole, and for this reason I gave the name */dev/joe vortex* to these circular patterns. There is also an optical illusion here which makes it possible to view it as a 3-dimensional shape, a bar of triangular cross-section which has been bent into a loop and twisted like a Möbius strip, although my impression of it is that it has a full twist, and so does not have the one-sidedness that comes with a Möbius strip. This illusion works better for some S,L pairs than for others.

In the large circle, the pattern looks like a set of rather oval dominoes which have been set up in a tight circle and toppled into one another. The way that each piece completely fills the concavity in the next piece creates this illusion. It's not always that way, but when the ratio of S:L is appropriate for filling a large portion of the area of the circles, it commonly is. Despite the very different appearance, it's the same tile used in both patterns.

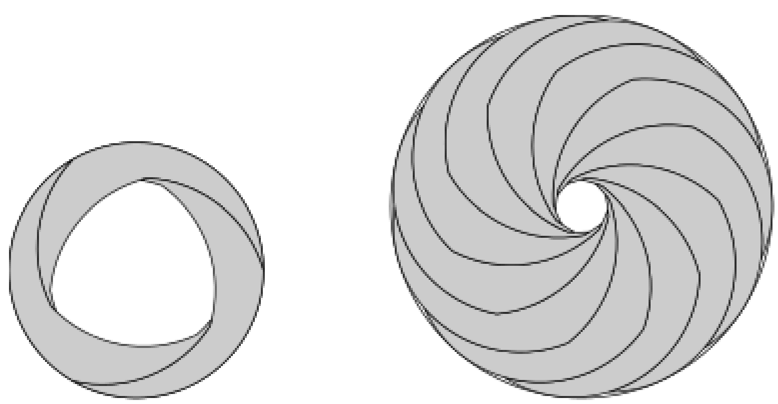

*Fig. 3: /dev/joe vortex of type 1 for S=3, L=11*

Since the areas of the two circles are in the ratio 4:9, and the goal was to fill as large a portion of the two circles as possible, the number of pieces used in the small and large circles should be close to this ratio in order to fill a large portion of the circle. However, there is nothing preventing the creation of vortices using numbers quite different from this ratio; it just creates a large hole in the middle of one of the circles, as in the case for S=3, L=11, shown in Figure 3.

1 http://www2.stetson.edu/~efriedma/mathmagic/0312.html

## Determination of the Shapes

The shape of a crescent is determined by the outside (which is always 1/S of a circle), the length of the common arc, and the angle between two arcs. In the larger circle, only a part of the outside arc is actually on the outside of the overall shape. Part of the outside arc lies against one of the common arcs from the trailing edge of another crescent. The portion of outside arc which can actually fit on the outside is determined by the overlap of L equally spaced circles of diameter 2, each internally tangent to the circle of diameter 3. The common arc is the difference between these two arc lengths, and can be determined purely as a function of S and L. A detailed derivation appears below.

The preceding construction also determines the sharp angle between the outside arc and the trailing edge. In the small circle, this sharp angle forms a smooth edge with the angle between the outside arc and the leading edge of another piece at the edge of the circle, and that is one of the common angles, so this sharp angle determines the common angle.

The shapes I have shown so far all have 6 edges, but this is not always the case for /dev/joe crescents. A crescent ends when the leading edge and trailing edge intersect. The final arc on each edge is cut short at this point. Two additional parameters, A and B, are defined as the number of arcs making up the leading and trailing edges, respectively. They aren't always 2 and 3. In typical cases which are good candidates for the area maximization of the original problem, usually A=S-1 and B=S.

## Special Cases

In some cases, as in the one shown in Figure 2, the hole in the center cannot be filled, because in the small circle it is bounded entirely by portions of the trailing edge, while in the large circle it is bounded entirely by portions of the leading edge. However, as seen above in Figure 3 for S=3, L=11, this is not always the case. When the last partial arc of trailing edge is longer than the last partial arc of leading edge, and B=A+1, both holes are bounded by trailing edge. When this happens, the solution can be improved by adding a small projection to each piece, such that the entire hole in the center of the large circle is filled. These bits will protrude into the much larger hole of the small circle, giving that hole a shape like a sawblade.

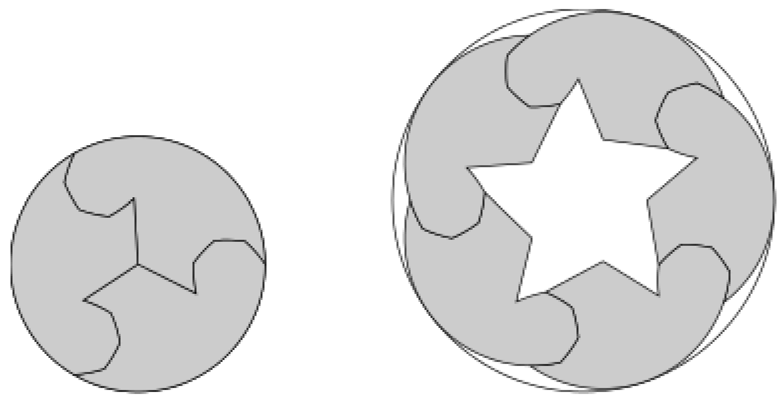

*Fig. 4: /dev/joe vortex of type 1 for S=3, L=5 (with A=B=4 and then connected to the center)*

At the other extreme in S:L, there are cases where the common edge is too short for the leading and trailing edges to intersect, and instead the edges curl back on themselves. In these cases, at an appropriate point the edges can be ended and simply connected straight to the middle of the small circle. This happens for S=3, L=5 as shown in Figure 4.

A very special case occurs when L=2S. This seems to cause A=B=S and for the leading and trailing edges to intersect in the center of the circle, with the final arcs bisecting one another, resulting in a completely filled small circle without any need to deviate from the standard design. When I tested the S=3, L=6 shape I got a shape which I think is identical to Bayly's solution for this case. Figure 5 shows this and the next two members of this family of solutions.

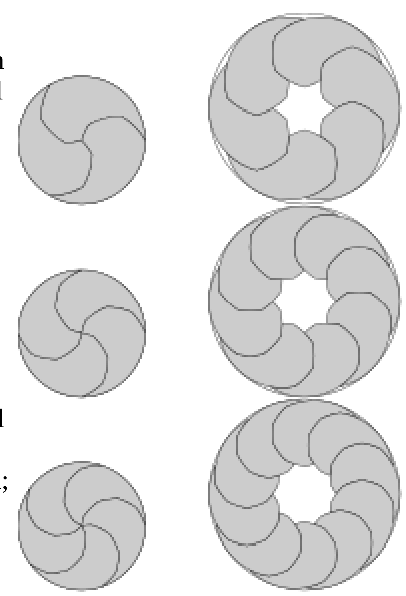

*Fig. 5: /dev/joe vortices of type 1 for L=2S, S=3 to 5*

## Type 2 Vortices

In the original problem (whose goal, remember, was to maximize the area covered), only the total number of pieces N=S+L was specified, not the individual values of S and L. However, each S,L pair has a maximum theoretical area which is reached when one of the circles is completely filled; either $N\pi/S$ if the small circle is filled, or $9N\pi/4L$ if the large one is filled. By the time I developed this strategy, there were already solutions for most cases up to N=16 which covered enough area that only the S,L pair with the largest value of this area limit even had a hope of matching that score. And in some cases my solutions did not measure up. When more of the larger circle was filled, I noticed that the central hole was sometimes quite tiny, and the area would not be large enough to reach the goal even if it was filled in. The problem was that I had made the outside be made of arcs of radius 1, and this left some little slivers of area along the edge of the circle which could never be filled.

This led me to develop a second type of crescent, one where the outside was an arc of radius 1.5. This allowed the slivers along the edge to be captured, while losing slivers along the edge of the small circle, which had area to spare.

It turns out that inscribing S arcs of radius 1.5 and length equal to 1/L of the large circle along the inside of the small circle does not completely fill the small circle for typical S:L ratios. The natural thing to do then is to space them equally and let the remaining space along the edge of the small circle become another edge of the shape. So the outside now consists of two arcs, one of radius 1.5 and one of radius 1. Since in the large circle, the arcs of radius 1.5 completely fill the edge, those arcs are always on the outside and never nest against another arc. Instead, the arcs of radius 1 become the common arcs that are repeated along the leading and trailing edges. This leads to solutions like the one shown in Figure 6.

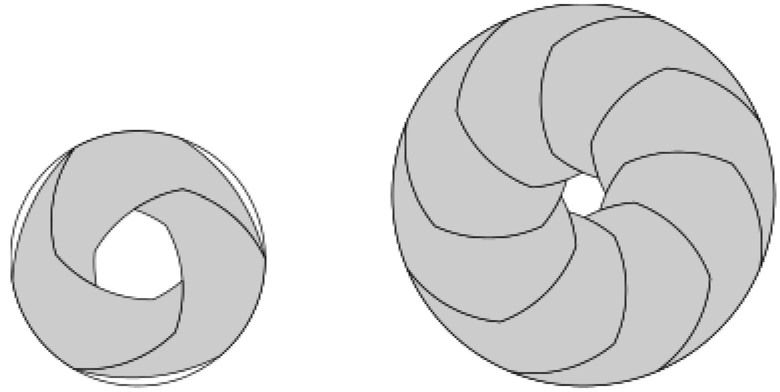

*Fig. 6: /dev/joe vortex of type 2 for S=3, L=8*

## Type 3 Vortices

It is possible that for some systems very close to the S:L ratio of 4:9, a hybrid approach in between these two may provide a better area. In this shape, part of the border is an arc of radius 1.5 but it is less than 1/L of a circle. This is a subject for further research and will not be covered here.

## Mathematical Basis of the Construction

I applied geometry and trigonometry to diagrams of the figures I was drawing to determine the placement of the various points necessary to draw these diagrams and determine their areas.

Begin by constructing a circle of radius 1.5 and a sector of that circle of angle $2\pi/L$ bisected by a horizontal radius of the circle. An angle $\pi/L$ appears above the horizontal and another $\pi/L$ below it. Now draw a sector of radius 1 and angle 2pi/S, tangent to the circle at its intersection with that same horizontal, and with its bottom point on the lower radius of the first sector. Necessarily, the distance between the centers is 1/2. The arc of radius 1 drawn here is the outside edge of the crescent. The part which lies inside the first sector is the part which lies on the outside when the crescents are placed in the larger circle; symmetry requires that this arc exactly fits in the sector of angle $2\pi/L$.

Let A be the intersection of this arc with the upper radius of the first sector. Draw another radius inside the second sector, to A. This divides the second sector into two pieces, an upper portion of angle $\varphi$ and a lower portion of angle $\theta$ which is itself bisected by the horizontal radius; $\varphi+\theta=2\pi/S$. In addition, define $\beta = (\pi-\varphi)/2$ as one of the equal angles in this triangle with edges 1, 1, d.

Construct a vertical line through A, and define x to be the distance along this vertical line from A to the horizontal radius, and y the distance from the center of the sector of radius 1 to the right angle just created. Construct a chord across the sector of angle phi; define d to be its length. This produces Figure 7.

Now we have $\sin(\theta/2) = x$ and $\tan(\pi/L) = x/(y+0.5)$ and also $x^2+y^2 = 1$. Solve the last equation to get $y = (1-x^2)^{1/2}$, and substitute it into the second equation:

$$\tan(\pi/L) = x/((1-x^2)^{1/2}+0.5)$$

$$x = \tan(\pi/L)((1-x^2)^{1/2}+0.5)$$

$$x = \tan(\pi/L)(1-x^2)^{1/2}+0.5\tan(\pi/L)$$

$$x-0.5\tan(\pi/L) = \tan(\pi/L)(1-x^2)^{1/2}$$

Now square both sides to eliminate the square root:

$$(x-0.5\tan(\pi/L))^2 = \tan^2(\pi/L)(1-x^2)$$

$$x^2+0.25\tan^2(\pi/L) - x\tan(\pi/L) = \tan^2(\pi/L) - x^2\tan^2(\pi/L)$$

$$x^2(1+\tan^2(\pi/L)) - x\tan(\pi/L) - 0.75\tan^2(\pi/L) = 0$$

Solve the quadratic for x:

$$x = (\tan(\pi/L) +/- (\tan^2(\pi/L)+3\tan^2(\pi/L)(1+\tan^2(\pi/L)))^{1/2})/(2+2\tan^2(\pi/L))$$

$$x = (\tan(\pi/L) +/- (4\tan^2(\pi/L)+3\tan^4(\pi/L))^{1/2})/(2+2\tan^2(\pi/L))$$

Pull a $\tan(\pi/L)$ out of the numerator:

$$x = (1 +/- (4+3\tan^2(\pi/L))^{1/2})\tan(\pi/L)/(2+2\tan^2(\pi/L))$$

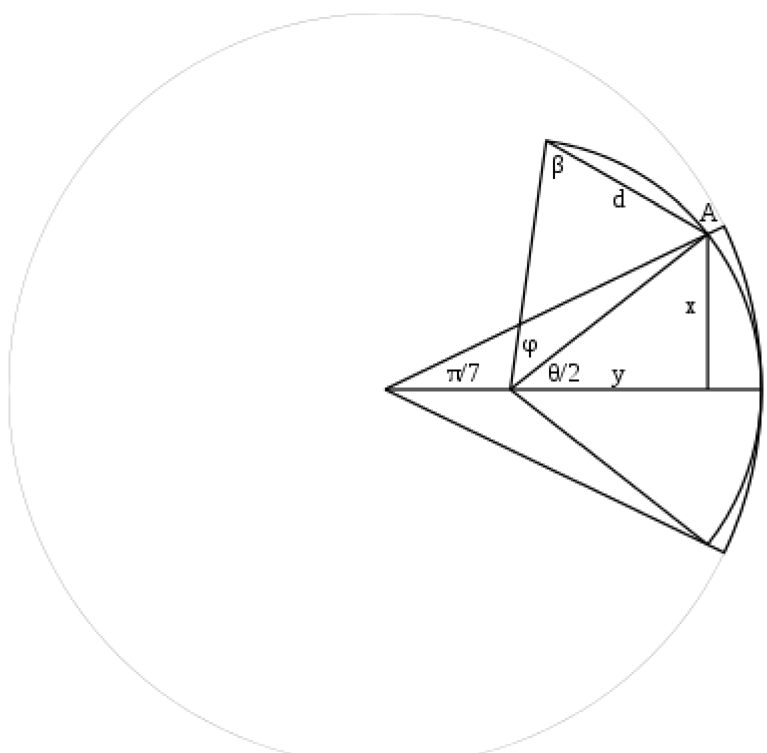


*Fig. 7: Construction used to determine length of outside and common arcs for Type 1 vortices. S=3 and L=7 for this example.*

As long as L is at least 3, the tangent has a positive value. For x to be positive, we need the first term to be positive, and $(4+3\tan^2(\pi/L))^{1/2} > 2$, so we need the positive choice of the +/- sign.

$x = (1 + (4+3\tan^2(\pi/L))^{1/2})\tan(\pi/L)/(2+2\tan^2(\pi/L))$

It's possible to apply other identities here but it doesn't really get much simpler than this. This form is useful in that it only requires calculating one trig function to numerically calculate x from L. Since $\sin(\theta/2) = x$ means $\theta = 2 \operatorname{asin}(x)$, we can calculate θ from L as well.

We also have $\varphi = 2\pi/S - \theta$ and $d = 2 \sin(\varphi/2)$.

We don't need x and y in the diagram any more. They were merely a means to calculate θ from L. Drop them (and the vertical line which was of length x), and duplicate the remaining lines rotated counterclockwise by 2π/L.

Connect the centers of the two sectors of radius 1 with a line segment. If this was repeated all the way around the circle, the segments like this would form a regular L-gon with circumradius 1/2. Let n be the length of one of these lines.

$n = 1/2\ (2 \sin(\pi/L)) = \sin(\pi/L)$

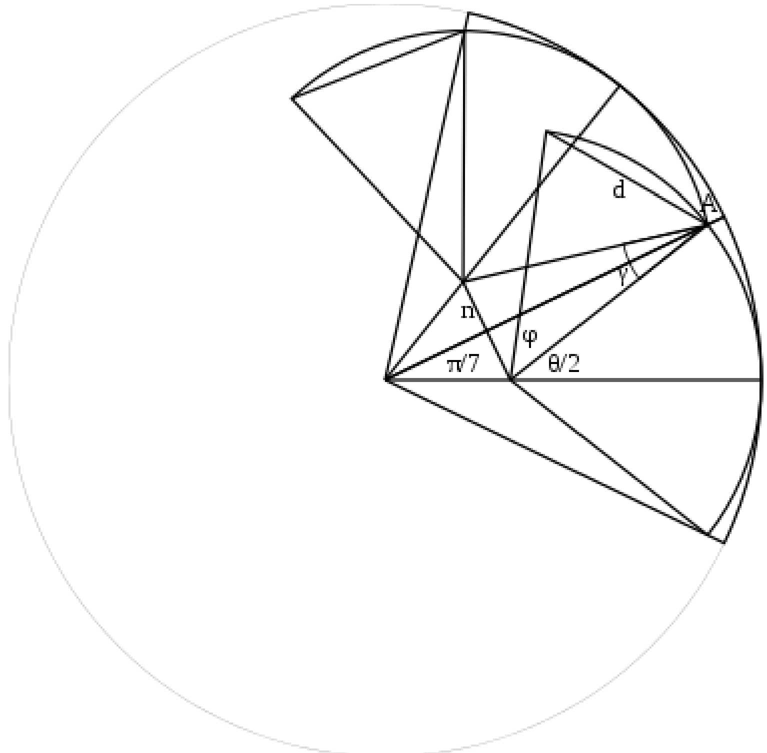

*Fig. 8: More of the construction*

Also, define γ as the angle at A in the triangle now formed with sides 1, 1, n. So $\gamma/2 = \operatorname{asin}(n/2)$ or $\gamma = 2 \operatorname{asin}(n/2)$. This yields Figure 8.

The next goal is to add another arc to the shape to the left of the arc with chord d. It will be congruent to that arc, and share one endpoint with it. The angle at which this arc is drawn must be determined.

I used the angle between chord d and the corresponding chord for the next arc. In order to do this, we need to take the drawing here and place it on the small circle, centered at the center of the small sector from the upper tile. Some superfluous lines have been removed, but we need to keep the entire upper tile and the arc with chord d, since it forms part of the trailing edge of our tile. This produces Figure 9.

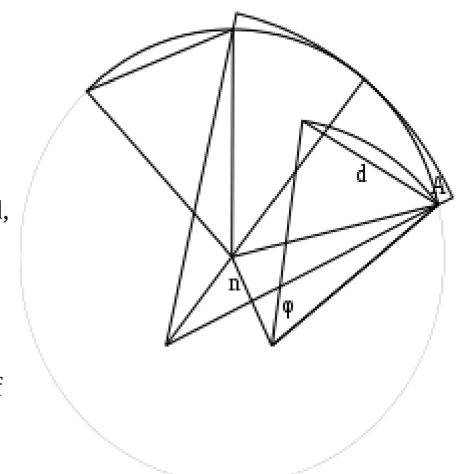

*Fig. 9: Moving to the small circle*

The next tile would be properly placed at the same center and rotated by 2π/S. See Figure 10. The arc with chord labeled d in the upper tile forms part of the leading edge of the lower tile. I used the angle between the two chords (d and its adjacent copy) to determine the placement of the succeeding arcs. This angle α is composed of two copies of the angle, β, between the chord and one of its radii minus the overlapping section, γ. The whole angle $\alpha = 2\beta - \gamma = \pi - \varphi - \gamma$.

Now we can dispense with the extra lines and just draw copies of this sector such that their chords form angles α until the leading and trailing edges meet. This is shown in Figure 11. It looks as if parts of three of these sectors and chords form a trapezoid, but it's a near miss and the two segments that appear to coincide to form the long edge actual intersect, form an angle of less than one degree.

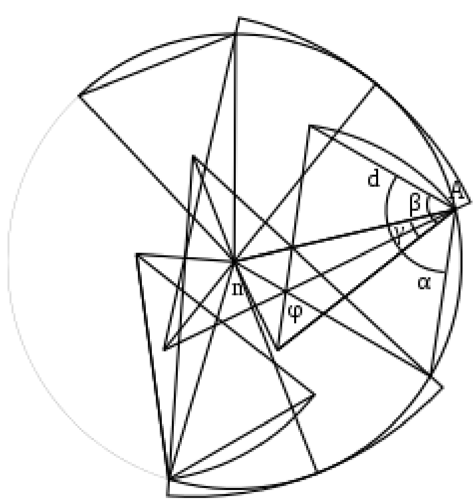

*Fig. 10: Two copies of the partial tile on the small circle.*

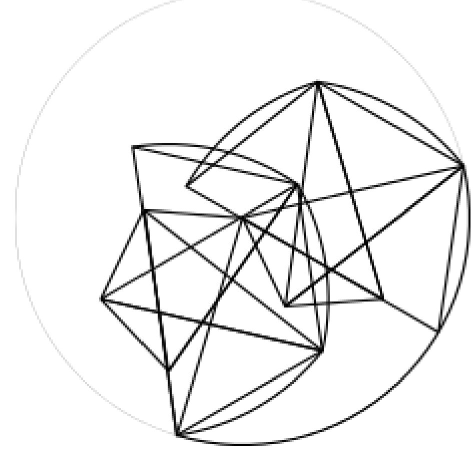
*Fig. 11: Completing the tile.*

## The Surprising Coincidence

In the previous construction, for S=3 and L=7, θ is some irrational fraction of a circle, and so are φ and γ. But α works out to exactly 13π/21. So if I continued drawing these arcs and chords until I had drawn 21 of them, they would then start to coincide after I've drawn 21 chords. These chords form the perfect star polygon {21/4} (Figure 12), that is, a polygon with 21 equal sides and 21 equal angles which loops 4 times around the center, with each vertex is connected to the 4th nearest one on either side.

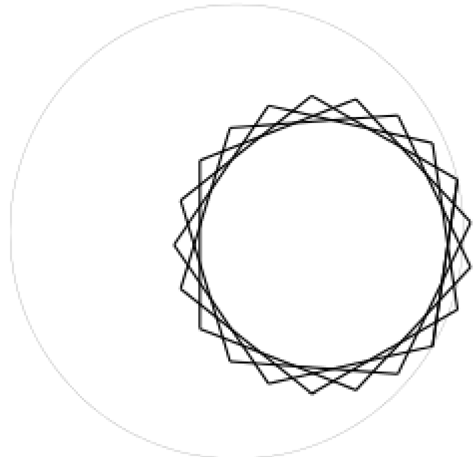

*Fig. 12: Star polygon {21/4} formed by iterating chords across common arcs of the S=3, L=7 tile well beyond the point needed for its leading and trailing edges to meet.*

Why is this? Why should these shapes lead to such an angle? It works for other cases, too. In general, the chords form a star polygon {SL/(L-S)} (with any common factors divided out). In the L=2S special case, the polygon is {SL/S} or a simple regular L-gon.

Proof of this angle: Go back to the diagram in the large circle which features the segment x, but extend it to length 2x all the way across the chord of angle $\theta$. Repeat this all the way around the large circle so that these segments form a regular L-gon. See Figure 13.

Each angle of this L-gon is composed of two copies of the angle shown here as $\eta$ which, due to the triangle it is in, is $(\pi-\theta)/2$, and one copy of $\gamma$. Since it's the interior angle of a regular L-gon, it measures $\pi - 2\pi/L$. So $\pi - 2\pi/L = \pi - \theta + \gamma$, or $\gamma = \theta - 2\pi/L$. But $\alpha = \pi - \varphi - \gamma$ and $\varphi = 2\pi/S - \theta$, so $\alpha = \pi - 2\pi/S + \theta - \theta + 2\pi/L = \pi - 2\pi(1/S-1/L) = \pi - 2\pi(L-S)/SL$. So SL copies of this angle add to $(SL - 2L + 2S)\pi$, which is exactly the angle we need to make the star polygon {SL/(L-S)}.

It's easier to see this if you take the supplementary angle $\pi-\alpha$, which equals $2\pi(L-S)/SL$, which is just enough to bend the polygon around L-S full revolutions after SL copies of the angle.

## The Arc-Through-the-Center Coincidence Proven

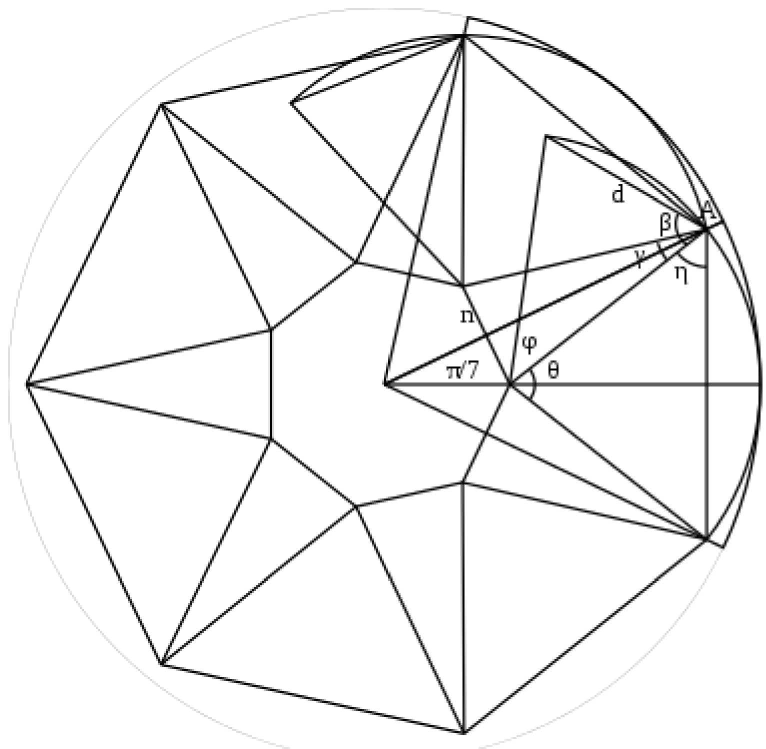


*Fig. 13: The L-gon of chords of length 2x.*

There is still the unexplained coincidence in the L=2S cases where the arc passes through the center of the small circle.

In these cases, the polygon has an even number of sides, and the distance of interest is the distance from the midpoint of one arc to the midpoint of the opposite arc. Aside from the conclusion I want to reach that this distance is 1, it's hard to measure this relative to any of the rest of the problem. But draw a segment bisecting each sector. These segments for two adjacent sectors intersect at some distance from the arc's midpoint (and another distance from the center about which the arc is drawn) which by symmetry is the same for both sectors. It's the same by symmetry for each pair of adjacent sectors, so it is the same for all of them, and all of them intersect in a single point, C. The distance from this C to the arc's midpoint B is half the distance of interest.

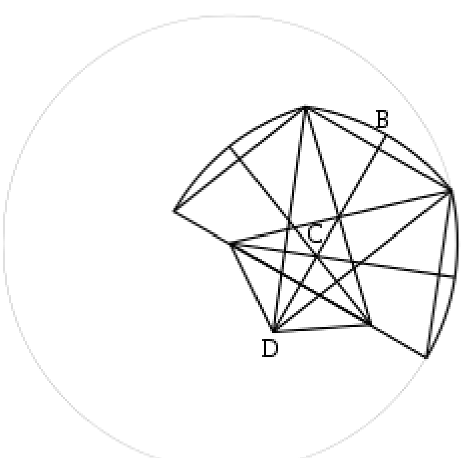


*Fig. 14: Bisecting the sectors*

To find the length of segment BC, I calculate the distance from C to the center D about which the arc is drawn, and subtract that from the known radius, 1. CD and its counterpart in the adjacent sector form two legs of an isosceles triangle, where the third side is the segment identified earlier as $n = \sin(\pi/L)$. These segments of length n form another star polygon similar to the one formed by the chords, so the angle at C is (L-S)/SL of a full revolution, or $2\pi(L-S)/SL$.

So $CD / (n/2) = \sin(\pi(L-S)/SL)$ or $CD = \sin(\pi(L-S)/SL) / 2\sin(\pi/L)$. In the special case where L=2S, $\sin(\pi(L-S)/SL) = \sin(\pi/L)$ so $CD = \sin(\pi/L) / 2\sin(\pi/L) = 1/2$. That makes BC also 1/2, and this is why the distance from arc midpoint to opposite arc midpoint is 1 in these cases.